Self-similar and translationally invariant structures in a bond-diluted Ising model

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LETTER TO THE EDITOR

Self-similar and translationally invariant structures in a bond-diluted Ising model

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Abstract. A bond-diluted Ising model is used to simulate a process from a Cantor set to a translationally invariant lattice. The free energy function shows that there is no singularity, which seems to imply that no transition exists in this process.

It has been proposed that a geometrical parameter, the lacunarity, of the fractal (Mandelbrot 1979, Gefen *et al* 1980, 1983, 1984) may be used to measure the extent of the failure of a fractal to be translationally invariant. Gefen *et al* (1984) and Lin and Yang (1986) have respectively suggested an approximate expression of lacunarity. They point out that for translationally invariant lattices the lacunarity should be zero. Therefore the relative value of the lacunarity may reflect the extent of deviation of a translationally invariant lattice: the larger the value of lacunarity, the larger its deviation from the periodic lattice. However, how can we realise that a fractal lattice continuously goes to a translationally invariant lattice? This is still an open problem. Moreover, this problem may be associated with some interesting physical processes, such as certain crystal growth processes.

Here we propose a special diluted magnetic model, the diluted Ising model, with which to simulate a process from the fractal lattice to the translationally invariant lattice. We expect that this descriptive method will be able to provide some instructive knowledge for some real physical processes.

We now consider a Cantor-like set, which is constructed in the following way: we start with a straight line. This straight line is divided into three segments, of which the middle segment is diluted instead of removed. The procedure is then repeated for the smaller segments and iterated until the microscopic length scale is reached. The resulting shape is self-similar all over the intermediate length scale. The first four stages of the construction of the Cantor-like set are shown in figure 1. The Ising spins are placed on each site of the Cantor-like set. We assume that the interactions are allowed between nearest neighbours only. There are two kinds of interaction parameter: J, for undiluted bonds and $J' = \alpha J$, for diluted bonds, where α is the 'dilution factor' which describes the extent to which bonds are diluted and α varies continuously, $0 \leq \alpha \leq 1$. In particular, there are two limits $\alpha = 0$ and $\alpha = 1$ which respectively correspond to the case of bonds removed, thus yielding a real Cantor set with fractal dimension $d_f = \ln 2/\ln 3$, and all bonds maintained, thus obtaining a one-dimensional translationally invariant lattice. In the absence of a field, the Ising Hamiltonian of the

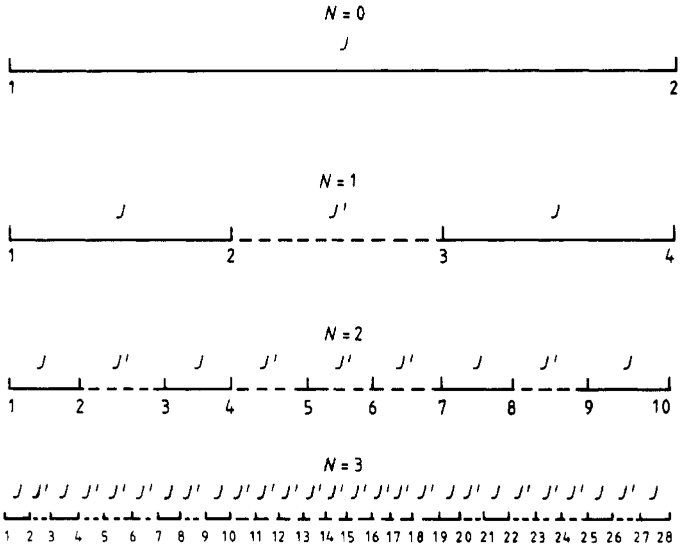

Figure 1. The first four stages of the Cantor-like set. The broken lines indicate diluted segments with interaction J' and full lines indicate undiluted parts with interaction J.

first three stages of the construction are written as

$$-\beta H_0 = J\sigma_1\sigma_2 \qquad (N=0) \tag{1}$$

$$-\beta H_1 = J(\sigma_1\sigma_2+\sigma_3\sigma_4)+J'\sigma_2\sigma_3 \qquad (N=1) \tag{2}$$

and

$$\begin{aligned}-\beta H_2 &= J(\sigma_1\sigma_2+\sigma_3\sigma_4+\sigma_7\sigma_8+\sigma_9\sigma_{10})+J'(\sigma_2\sigma_3+\sigma_4\sigma_5+\sigma_5\sigma_6+\sigma_6\sigma_7+\sigma_8\sigma_9)\\ &= J(\sigma_{3^{N-2}}\sigma_{3^{N-2}+1}+\sigma_{2\times 3^{N-2}+1}\sigma_{3^{N-1}+1}+\sigma_{2\times 3^{N-1}+1}\sigma_{(2\times 3^{N-1}+1)+3^{N-2}}\\ &\quad+\sigma_{(2\times 3^{N-1}+1)+2\times 3^{N-2}}\sigma_{3^N+1})\\ &\quad+J'(\sigma_{3^{N-2}+1}\sigma_{2\times 3^{N-2}+1}+\sum_{3^{N-1}+1}^{2\times 3^{N-1}}\sigma_i\sigma_{i+1}\\ &\quad+\sigma_{(2\times 3^{N-1}+1)+3^{N-2}}\sigma_{(2\times 3^{N-1}+1)+2\times 3^{N-2}}) \qquad (N=2)\end{aligned} \tag{3}$$

where N denotes the Nth step of construction and we assume that the periodic boundary condition in each stage of the construction is used. Due to the complexity we will not give the general expression of the Ising Hamiltonian; the above expression will suffice for our purpose. Some thermodynamic functions of the Ising model, such as free energy, may be calculated on our construction by using the transfer matrix technique. Let us first of all define the matrix elements as follows:

$$\langle\sigma_i|R|\sigma_{i+1}\rangle = \exp(J\sigma_i\sigma_{i+1}). \tag{4}$$

and

$$\langle\sigma_i|P|\sigma_{i+i}\rangle = \exp(J'\sigma_i\sigma_{i+1}). \tag{5}$$

Obviously, these matrices commute with one another. The partition function of the first three stages of the construction are given by

$$Z(H_0) = \sum \exp(J\sigma_1\sigma_2) = \text{Tr } R \tag{6}$$

$$Z(H_1) = \sum \exp[J(\sigma_1\sigma_2 + \sigma_3\sigma_4) + J'\sigma_2\sigma_3] = \text{Tr } R^2P \tag{7}$$

and

$$Z(H_2) = \sum \exp[J(\sigma_1\sigma_2 + \sigma_3\sigma_4 + \sigma_7\sigma_8 + \sigma_9\sigma_{10}) + J'(\sigma_2\sigma_3 + \sigma_4\sigma_5 + \sigma_5\sigma_6 + \sigma_6\sigma_7 + \sigma_8\sigma_9)]$$
$$= \text{Tr}(R^4p^5). \tag{8}$$

In general, we have

$$Z(H_N) = \text{Tr}(R^KP^L) \tag{9}$$

where $K = 2^N$ and $L = 3^N - 2^N$. In each stage of the construction the partition function may be obtained from that of the previous stage; by replacing R by RPR and P by P^3, for example, from $Z(H_1) = \text{Tr}(R^2P)$, we obtain $Z(H_2) = \text{Tr}[(RPR)^2P^3] = \text{Tr}(R^4P^5)$.

Due to being matrices, R and P commute and can be diagonalised simultaneously. The partition function of H_N may be written as

$$Z(H_N) = \text{Tr}(R^KP^L) = \lambda_1^K\eta_1^L + \lambda_2^K\eta_2^L \tag{10}$$

where λ_1 and λ_2 are eigenvalues of the matrix R and they are respectively

$$\lambda_1 = 2\cosh J \tag{11}$$

and

$$\lambda_2 = 2\sinh J \tag{12}$$

and η_1 and η_2 are eigenvalues of the matrix P and they are given by

$$\eta_1 = 2\cosh J' = 2\cosh(\alpha J) \tag{13}$$

and

$$\eta_2 = 2\sinh J' = 2\sinh(\alpha J). \tag{14}$$

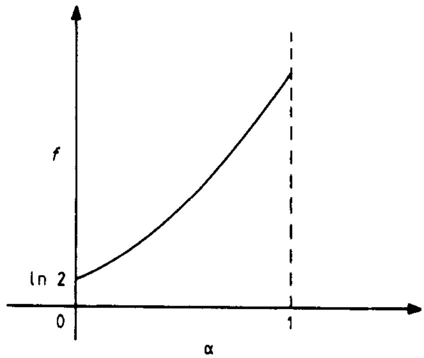

Figure 2. Plot of free energy against dilution factor α. Temperature T and exchange parameter J are given.

In the thermodynamic limit the free energy per site is given by

$$\begin{aligned} f &= \lim_{N\to\infty} (-kT/3^N) \ln Z(H_N) \\ &= -kT\{\ln \eta_1 + \lim_{N\to\infty} (3^N)^{-1} \ln[1+(\tanh J)^{2^N}(\tanh \alpha J)^{3^N-2^N}]\}. \end{aligned} \qquad (15)$$

In the special case of $\alpha = 0$, $f = -kT \ln 2$. This is the free energy on the Cantor set. It is noted that f is independent of J. This can be explained by the fact that the Cantor set consists of isolated sites only. In the $\alpha = 1$ case we have $f = -kT \ln(2 \cosh J)$ which is simply the result obtained for the one-dimensional translationally invariant lattice. Figure 2 is a plot of the free energy against the dilution factor α for given temperature T and exchange parameter J. The fact that no singularity exists in expression (15) seems to imply that no transition exists when a process develops from a self-similar structure to a translationally invariant structure. This topic deserves a full study.

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